Longitudinal cooling of non-neutral plasma by energy exchange

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The optimal values of Q and $\Delta\omega$ ($\Delta\omega \equiv \omega - \Omega$) for cooling a pure electron plasma with a microwave bath have been calculated. An electron plasma, which has no internal degree of freedom, cannot be cooled below the temperature of a heat bath. However, longitudinal cooling can be achieved by energy transfer from the poorly cooled longitudinal degree of freedom to the well-cooled (by synchrotron radiation) transverse degree of freedom. To do this, a microwave bath is introduced to the electron plasma. A microwave tuned to a frequency below the gyrofrequency forces electrons moving towards the microwave to absorb a microwave photon. The electrons move up one in Landau state and then lose their longitudinal momenta. In this process, the longitudinal temperature of the electron plasma decreases. On the basis that the perpendicular temperature is below the Landau temperature of the plasma, we set up two level transition equations and then derive a Fokker-Planck equation from them. With the aid of a finite element method (FEM) code for the equation, the cooling times for several values of the magnetic field, the microwave cavity (Q), and the relative detuning frequency from the gyrofrequency ($\Delta\omega$) are calculated. Thus optimal values of the microwave cavity and the detuning frequency for longitudinal cooling of a strongly magnetized electron plasma with a microwave bath have been found. By applying these optimal values with an appropriate microwave intensity, the best cooling can be obtained. For an electron plasma magnetized to 10 T, the cooling time to the solid state is approximately two hours.

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I. INTRODUCTION

The concept of crystalline non-neutral plasma, regarded as a new state of matter, has been studied for a variety of fundamental and applied physics areas, including the study of space-charge-dominated beams, the study of Coulomb crystals, the realization of high luminosity ion colliders, the application to ultrahigh resolution nuclear experiments and to the atomic physics research, etc. Crystallization occurs as non-neutral plasmas and beams are cooled below the transition temperature. In fact, as seen in many Penning trap experiments, the non-neutral plasmas has three different phases: fluid, fcc, and bcc [1]. Crystallization in one dimension has been observed in the beams at the Aarhus accelerator [2], in agreement with calculations [3], and crystallization in three dimension has been observed in the ion Penning trap at NIST [4] and in dusty plasmas [5].

In high energy physics, Penning traps and antiparticle storage rings have been used for experimental tests of the *CPT* theorem [6,7], which predicts equivalence of various physical parameters such as masses, charge-to-mass ratio, magnetic moments, and gyromagnetic ratio for particles and antiparticles [8]. Charged particles can be confined perfectly in an ideal cylindrically symmetric trap with a uniform axial magnetic field [9,10], which is the basic setup of the Penning trap [11]. This approach, the use of Penning trap, has been favored and widely used because the particle can be cooled down to a temperature of the order of 10 mK. Penning traps at CERN have been used to capture antiparticles for high-resolution measurements for proton mass and for mass spectrometry of nuclei [12].

Laser cooling [13–15] has been the primary approach towards obtaining ultra cool beams [16,17] and plasmas [18,19]. A laser that is tuned to a frequency below the resonant frequency of the ion is directed at the ionic plasma. Ions moving towards the laser beams see an up-shifted laser beam, and thus can absorb the light. Subsequently they spontaneously emit a photon isotropically. Thus, in the full process, they lose momentum by recoil. This leads to cooling. Such a method naturally works only for ions, not electrons or protons, as they have the internal resonances needed for narrow absorption. For non-ionic beams, electron cooling has been used, but such cooling has not produced ultracool beams [20].

In the case of a strongly magnetized plasma, the cyclotron frequency is much larger than the plasma frequency which is the main frequency of the longitudinal oscillation. The fast gyromotion compared to the longitudinal motion implies that the total action of the gyromotion is an adiabatic invariant [21]. The existence of the invariant promises that the longitudinal and transverse temperatures can be well-defined separately in the system and that all the thermodynamic potentials should be the functions of the two temperatures [22].

For this reason we already investigated the phase transition of strongly magnetized electron plasmas in Penning traps, and we concluded that the phase transition can occur on the condition that longitudinal temperature is below a certain value irrespective of transverse temperature [23]. Now the question is how to decrease the longitudinal temperature to the critical value. We suggest a microwave cooling method as one of the possible ways. Applying a tuned microwave into the longitudinal direction, the longitudinal energy can be reduced and then the temperature can be

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dropped down below the critical value. This means that the electron plasma crystallization can be achieved. In the following section we briefly explain the thermodynamics and the guiding center dynamics of the strongly magnetized plasmas, and show how their phase responds to the longitudinal temperature. In the next section we suggest how to reduce the longitudinal temperature and the result is shown.

II. THERMAL EQUILIBRIUM OF STRONGLY MAGNETIZED PLASMA

Consider a strongly magnetized nonneutral plasma as an ideal system of mobile particles of charge Q, number density n_e, and temperature k_BT, immersed in a heat bath. Using the Wigner-Seitz radius $a\equiv(3/4\pi n_e)^{1/3}$ and $\sqrt{3}\omega_p^{-1}$ ($\omega_p \equiv \sqrt{4\pi n_e Q^2/m}$) as the units of length and time, the thermodynamics of the nonneutral plasma can be described in terms of several dimensionless parameters, including the Coulomb coupling parameter,

$$\Gamma = \frac{Q^2}{ak_BT}, \tag{1}$$

which is roughly the ratio of the Coulomb potential energy to the thermal energy per particle. From the definition of the Coulomb coupling parameter, $1/\Gamma$ can be interpreted as the temperature in the unit of Coulomb energy, Q^2/a.

From theoretical works on unmagnetized plasmas and the results of Penning trap experiments, it is known that one component plasma has two phases, bcc (crystalline) and fluid, and the phase transition occurs at $\Gamma \approx 170$ [24,25]. In theoretical studies, one of the approaches to understand the nature of the plasma crystal is molecular dynamics simulation (MD simulation). Hamaguchi and Farouki found the phase transition in the unmagnetized plasma system with the MD simulation. Also, they showed that the plasma has two different phases-bcc and fluid [1,26], as seen in many Penning trap experiments. In Penning trap experiments, it is shown that the ion crystallization (bcc phase) can be achieved with a sufficiently strong 3D cooling force. Laser cooling, which is supposed to be one of the most promising methods to obtain the crystalline plasma, has been used to achieve the plasma temperature in the mK range. But despite considerable success in crystallization of ionic plasmas [4], there has been no successful experiment with electron plasmas, because laser cooling is ineffective in cooling nonionic plasmas.

In the system of a strongly magnetized electron plasma, the strong magnetic field forces the transverse motion to separate into gyromotion, the action of which is an adiabatic invariant, and into the $E\times B$ drift motion, which is slower by the order of a small adiabatic parameter, ω_p/Ω. When the transverse temperature is sufficiently low, such that $k_BT_\perp < \hbar\Omega$, the gyromotion energy is quantized as $E_{\perp,n}=\hbar\Omega\left(n+\frac{1}{2}\right)$ for the nonnegative integer n. As a result of the quantum treatment, the rest of the motions are described by the guiding center approximation. The Hamiltonian of an electron in the system reduces to

$$H = \hbar\Omega\left(\hat{a}_\perp^\dagger \hat{a}_\perp + \frac{1}{2}\right) + \frac{p_z^2}{2m} + \Phi(X,Y,z), \tag{2}$$

where the potential $\Phi(X,Y,z)$ is the sum of the pair potentials between an electron and the other electrons, and X and Y represent the guiding center position of the electron. A reduced Hamiltonian, excluding the quantized gyromotion,

$$H_\parallel = \frac{p_z^2}{2m} + \Phi(X,Y,z), \tag{3}$$

allows longitudinal dynamics and guiding center dynamics in the system. In this case, the guiding center positions, X and Y, are conjugate to each other, so that the reduced motion of the system can be described by two pairs of conjugate variables, (z,p_z) and (X,Y). In the dimensionless units of length, time, and energy, the equations of motion in the classical regime are

$$\frac{dz}{dt} = \frac{dH_\parallel}{dp_z}, \tag{4a}$$

$$\frac{dp_z}{dt} = -\frac{dH_\parallel}{dz}, \tag{4b}$$

$$\frac{dX}{dt} = -\frac{\omega_p}{\sqrt{3}\Omega}\frac{dH_\parallel}{dY}, \tag{4c}$$

$$\frac{dY}{dt} = \frac{\omega_p}{\sqrt{3}\Omega}\frac{dH_\parallel}{dX}, \tag{4d}$$

where the guiding center motion is clearly slower than the longitudinal motion by the order of ω_p/Ω.

An electron plasma to which a uniform magnetic field is applied usually has two characteristic temperatures, the transverse and the longitudinal, and thus has an anisotropic distribution initially. The collisions between electrons force the two temperatures to equilibrate in a characteristic time scale. The equilibration rate, the inverse of the equilibration time, is determined by the number of collisions per unit time, and the rate, ν_c, can be written as

$$\nu_c = n\bar{v}\bar{b}^2 I(1/\epsilon), \tag{5}$$

where $\bar{v}=\sqrt{2k_BT_\parallel/m}$ is the standard deviation for the distribution, $\bar{b}=2e^2/k_BT_\parallel$ is twice the classical distance of closest approach, and $\epsilon=\bar{v}/\bar{b}\Omega$ is a measure of magnetic field strength. For a weakly magnetized plasma, $(\epsilon \gg 1)$, $I(1/\epsilon)$ is obtained as $-(\sqrt{2\pi}/15)\ln(1/3\epsilon)$ [22], exactly as for unmagnetized plasma, except that $1/3\epsilon$ replaces $\lambda_D/\bar{b}$ in the logarithm, where $\lambda_D=(k_BT/4\pi ne^2)^{1/2}$ is the Debye length. The logarithm dependency of ν_c for the weakly magnetized plasma creates a high equilibration rate, so that the two temperatures equilibrate quickly. This is because a large radius of cyclotron motion will generate strong Coulomb collisions among electrons, and especially when the radius is large compared to the Debye length, the Coulomb collisions over the range of the Debye sphere will be replaced by the collisions of the gyromotion. In this case, we cannot apply the guiding center approximation to the weakly magnetized plasma, and we may assume that the plasma has an isotropic velocity distribution initially.

However, in the case of strongly magnetized plasma ($\epsilon \ll 1$), the small radius of the gyromotion will never generate strong Coulomb collisions among electrons. Instead, the weak Coulomb collisions will cause negligible exchange of energy between the longitudinal and perpendicular degrees of freedom, and thus the longitudinal and perpendicular motions will need to be described separately for the energy-exchanging time scale. The transverse motion would be characterized as the gyrofrequency, and the longitudinal motion characterized as two frequencies, $\omega_\| = \overline{v}/\overline{b}$ and ω_p. Since the two longitudinal frequencies are much smaller than the transverse frequency, an adiabatic invariant of the gyromotion, which is the cyclotron action, is well conserved, and also, the longitudinal temperature does not necessarily equal the perpendicular temperature.

Due to the adiabatic invariant of the gyromotion for a strongly magnetized plasma, the equilibration rate between the transverse and longitudinal temperatures is an exponentially small function of $-1/\epsilon_p$ and $-1/\epsilon$, where $\epsilon_p = \omega_p/\Omega$ and $\epsilon = \omega_\|/\Omega$ are much smaller than unity [21,22]. The exponentially small exchange rate of the transverse cyclotron and the longitudinal energy, ν_c, prevents the two temperatures from relaxing to a common value over a long time scale $1/\nu_c$ [22,27]. On the time scale during which the transverse action is conserved, the distribution of the system can have the form

$$P = Z^{-1}\exp\left[-\frac{\hbar\Omega\left(n+\frac{1}{2}\right)}{k_B T_\perp} - \frac{\frac{p_z^2}{2m} + \Phi(X,Y,z)}{k_B T_\|}\right], \qquad (6)$$

where Z is the partition function of the system and $T_\|$ is the longitudinal temperature, defined as the kinetic temperature. Therefore, the thermodynamics of the plasma depends on both the transverse and the longitudinal temperatures. Especially when the transverse temperature is lower than $\hbar\Omega$, the longitudinal temperature will be the dominant parameter to describe the thermodynamics and the phase transition. From our molecular dynamics simulations for strongly magnetized plasma, we know that the phase transition from bcc to fluid occurs at $\Gamma_\| \approx 170$ [23]. Moreover, the existence of the transverse quantum structure suggests that the longitudinal temperature can be decreased by a longitudinal microwave that is well-tuned to the Doppler-shifted resonance frequency between the two nearest Landau levels.

III. TWO LEVEL TRANSITION EQUATION FOR MICROWAVE COOLING

A. The basic phenomenon for microwave cooling

In this section, we will consider the two level system of a transversely quantized electron interacting with a pair of oppositely traveling waves, and present an argument which gives a picture of the derivation of the dissipative equation for the electron.

We consider a simplified electron system consisting of two level, separated by the energy $\hbar\Omega$. This is acted upon by an ideal microwave field of angular frequency, ω, traveling along magnetic field direction. The microwave photon causes resonant transfer of population between the two levels when the longitudinal velocity of the electron causes a Doppler shift that compensates the detuning of the microwave from the resonance. This really happens when

$$\frac{p_z}{m} = \frac{\Omega - \omega}{k_0}, \qquad (7)$$

where $k_0 = \Omega/c$ is the wave vector of the microwave photon. When the electron absorbs $\hbar\omega$ from the microwave field, it must also compensate for the loss of momentum. Thus, the electron loses an additional momentum, $\hbar k_0$, along the direction of propagation of the microwave. Consequently the electron loses its longitudinal energy by losing its longitudinal momentum. In the classical point of view, this process can be described by a Fokker-Planck equation. The Fokker-Planck equation is well known as

$$\frac{\partial f(p_z)}{\partial t} = \frac{\partial}{\partial p_z}\left[\nu_s(p_z)f(p_z) + D_s(p_z)\frac{\partial}{\partial p_z}f(p_z)\right], \qquad (8)$$

where $\nu_s(p_z)$ and $D_s(p_z)$ are the dissipative and the diffusion coefficient, respectively. The dissipative coefficient of the Fokker-Planck equation decides a cooling time scale of the system.

Let us proceed to find a proper equation for the entire cooling process. Since an electron interacting with a microwave field loses $\hbar k_0$ from its longitudinal momentum for the microwave transition time, the dissipative coefficient can simply be proportional to $\hbar k_0 W_s$ in which W_s is the microwave transition rate. Considering the portion of the electron plasma losing the longitudinal momentum as $C_p(n_{01})$ ($C_p(n_{01}) < 1$), we establish the equation as

$$\frac{\partial f(p_z)}{\partial t} \approx \hbar k_0 \frac{\partial}{\partial p_z}[C_p(n_{01})\{W_+(p_z) - W_-(p_z)\}f(p_z)], \qquad (9)$$

where $C_p(n_{01})$ is a function of the total number ratio between the ground and the first excited state, n_{01}, and the stimulated transition rate by the microwave, $W_s(p_z)$, is defined as

$$W_s(p_z) = F\frac{(\gamma/2)^2}{(\gamma/2)^2 + (\omega - \Omega + sk_0 p_z/m)^2}. \qquad (10)$$

In Eq. (10), F has $1/\mathrm{sec}$ as its unit. Usually, F can be expressed as

$$F = \frac{I\sigma}{\hbar\Omega}, \qquad (11)$$

where $I\sigma$ is the power of the microwave.

When the longitudinal velocity is small, Eq. (22) assumes the dissipative form,

$$\frac{\partial f(p)}{\partial t} \approx -\nu_s(n_{01})\frac{\partial}{\partial p_z}[p_z f(p_z)], \qquad (12)$$

where the dissipative coefficient, $\nu_s(n_{01})$, is

$$\nu_s(n_{01}) \approx C_p(n_{01}) \frac{4\hbar k_0^2 F(\gamma/2)^2(\omega-\Omega)}{m[(\gamma/2)^2+(\omega-\Omega)^2]^2}$$
$$= C_p(n_{01}) W_s(0) \frac{4\hbar k_0^2(\omega-\Omega)}{m[(\gamma/2)^2+(\omega-\Omega)^2]}. \quad (13)$$

The number ratio, n_{01}, should is decided by the ratio between the stimulated transition rate and the spontaneous transition rate, which is generated by the gyromotion. This means that the ratio between the stimulated transition rate and the spontaneous transition rate is also a function of the number ratio,

$$\frac{W_s(0)}{S} = r(n_{01}), \quad (14)$$

where the spontaneous transition rate, S, is

$$S \approx \frac{2e^2\Omega^2}{3mc^3}. \quad (15)$$

As a result of this, the dissipative coefficient can be expressed as

$$\nu_s(n_{01}) \approx R_p(n_{01}) S \frac{4\hbar k_0^2(\omega-\Omega)}{m[(\gamma/2)^2+(\omega-\Omega)^2]}, \quad (16)$$

where a new function, $R_p(n_{01})$, is simply $C_p(n_{01})r(n_{01})$. For a small number ratio ($n_{01}\sim 0.2$), the function, $r(n_{01})$, is approximately the same to unity ($r(n_{01})\sim 1$). The detuning frequency, $\omega-\Omega$, has the same order to γ. From these considerations, the cooling time scale is approximately

$$\tau_s = \frac{1}{2\nu_s \log_{10} e} \approx \frac{\gamma}{S \log_{10} e} \frac{m}{\hbar k_0^2}. \quad (17)$$

For 10 T as its magnetic field strength, the cooling time scale for the longitudinal cooling is about 1 h. Consequently the cooling time for an electron plasma to reach a critical temperature is about a few hours.

B. Two-level transition equation

In this section, we will obtain the basic master equations for microwave cooling of an electron plasma for the two lowest Landau levels. In the low transverse temperature limit ($k_B T_\perp < \hbar\Omega$), while the longitudinal temperature is much higher than the harmonic oscillation energy generated by plasma oscillation, ($k_B T_\parallel \gg \hbar\omega_p$), the longitudinal energy can be reduced by microwave radiation. Absorption of a microwave photon by an electron, in a Penning trap, moving it up one in Landau state, can reduce the longitudinal energy, just as laser cooling does for ionic plasma and equilibrium. The spontaneous radiation reduces the transverse energy, so that the transverse state moves back to the original Landau state until the transverse temperature is the same as the heat bath temperature. During the entire process, only the longitudinal temperature will decrease.

Let $f_n(p_z)$ denote the longitudinal momentum distribution of transverse quantum number n, which implies that the electrons are found in the nth Landau state with longitudinal momentum p_z at a certain time. Then $f_n(p_z)$ tends to increase with time, when electrons in other states make transitions to the state, and it tends to decrease with time, when the electrons in the state make transitions to other states. The longitudinal velocity distributions during the process can be described by a master equation,

$$\frac{\partial f_l(p)}{\partial t} = \sum_m \int dq[f_m(q)T_{ml}(q,p) - f_l(p)T_{lm}(p,q)], \quad (18)$$

where $T_{lm}(p,q)$ is the transition rate from lth to mth state.

Following the assumption that most of the particles are in the ground or first excited state, the two transition rates, $T_{10}(p_z',p_z)$ and $T_{01}(p_z,p_z')$ can be expressed as $\tilde{D}(p_z',p_z)$ and $\tilde{W}(p_z,p_z')$, respectively. The assumption leads the two master equations to

$$\frac{\partial f_0(p_z)}{\partial t} = \int dp_z' f_1(p_z')\tilde{D}(p_z',p_z) - f_0(p_z)\int dp_z' \tilde{W}(p_z,p_z'), \quad (19a)$$

$$\frac{\partial f_1(p_z')}{\partial t} = \int dp_z f_0(p_z)\tilde{W}(p_z,p_z') - f_1(p_z')\int dp_z \tilde{D}(p_z',p_z), \quad (19b)$$

where $\tilde{D}(p_z,p_z')$ represents the transition rate from the excited state to the ground, and $\tilde{W}(p_z,p_z')$ represents from the ground state to the excited.

An electron may decay from the first excited state to the ground by spontaneous and by microwave-stimulated emission. $\tilde{D}(p_z,p_z')$ may be separated into the spontaneous and the stimulated emission rate. We define the spontaneous emission rate as $\tilde{D}_0(p_z,p_z')$, and the spontaneous absorption rate as $\tilde{W}_0(p_z,p_z')$. In quantum mechanics, spontaneous decay can be interpreted as the interaction between an electron and the vacuum fluctuation of the electromagnetic field. An electron loses its energy during the decay, while the vacuum electromagnetic field gains the same amount of energy that the electron loses. The total Hamiltonian to describe this kind of system is expressed as the sum of the Hamiltonian of the electron, the Hamiltonian of the vacuum field, and the interacting Hamiltonian between the electron and the vacuum field. The total Hamiltonian is

$$H = H_0 + H_f + H_-, \quad (20)$$

where H_f is the Hamiltonian of the vacuum field, and H_- is the interacting Hamiltonian. Without loss of generality, the Hamiltonian of the electron in a constant magnetic field can be

$$H_0 = \frac{p_\perp^2}{2m} + \frac{m\Omega^2 r_\perp^2}{2} + \frac{p_z^2}{2m}, \quad (21)$$

where $r_\perp$ is the radial distance from the guiding center of the electron, and $p_\perp$ is its momentum. The interacting Hamiltonian, H_-, creates the spontaneous decay between two nearest Landau state of the electron. Thus, the spontaneous emission rate can be calculated from the famous Fermi golden rule,

$$A_{n',n}(p_z',p_z) = \frac{2\pi}{\hbar}\int |\langle f|H_-|i\rangle|^2 g(E)d\Omega_s, \tag{22}$$

where Ω_s is the solid angle of the emitted photon. The interacting Hamiltonian, H_-, is given by

$$H_- = \frac{e}{mc}\mathcal{A}_+\cdot\mathbf{p}, \tag{23}$$

where $\mathcal{A}_+$ is a creation operator that creates a photon on vacuum field. The initial state of the electron and the electromagnetic field is

$$|i\rangle = |n,p_z\rangle \otimes |0\rangle, \tag{24}$$

and the final state is

$$|f\rangle = |n',p_z'\rangle \otimes |\mathbf{k}\rangle. \tag{25}$$

Here, n' and p_z' in the first ket of the direct product represent the quantum number of the transverse Landau state and longitudinal momentum of the electron, respectively. The second term of the direct product represents the quantum state of the electromagnetic field. So, $\langle f|H_-|i\rangle$ is the emission element matrix between nth and n'th Landau state. Obviously, the n'th state is the lower Landau state, because the electromagnetic field gains energy as much as the the initial Landau state loses. The density of states in Eq. (22) is

$$g(E) = \frac{\omega^2 V}{\hbar(2\pi c)^3}, \tag{26}$$

where V is the volume of the system. Considering the transition between two nearest states ($n'=n-1$) as the main transition in our theory, the emission matrix element is

$$\langle f|H_-|i\rangle = \frac{e}{mc}\langle n',p_z'|\langle\mathbf{k}|\mathcal{A}_+|0\rangle\cdot\mathbf{p}|n,p_z\rangle. \tag{27}$$

The transition matrix of the field is

$$\langle\mathbf{k}|\mathcal{A}_+|0\rangle = \left(\frac{2\pi\hbar c^2}{\Omega V}\right)^{1/2}\hat{\mathbf{a}}e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{28}$$

where $\hat{\mathbf{a}}$ is a unit polarization vector that is normal to the propagation vector of the emitted photon. Applying a dipole approximation that the transverse wavelength of the field is far longer that the oscillation dimension and

$$[\mathbf{r},H_0] = i\hbar\mathbf{p}/m, \tag{29}$$

where H_0 is the Hamiltonian for only electron, we obtain the emission matrix element,

$$\langle f|H_-|i\rangle \approx ie(2\pi\hbar\Omega)^{1/2}\langle n',p_z''|\hat{\mathbf{a}}\cdot\mathbf{r}|n,p_z\rangle, \tag{30}$$

where $p_z''=p_z'-\hbar k_\parallel$ is the momentum, including the recoil effect. Combining these results, the probability is

$$A_{n',n}(p_z',p_z) \approx \frac{4e^2\omega^3}{3\hbar c^3(1+v_z/c)}|\langle n',p_z''|\mathbf{r}|n,p_z\rangle|^2, \tag{31}$$

which is the modified cyclotron radiation rate with a condition of longitudinal momentum change. Inserting the solution for an electron in a uniform magnetic field,

$$\langle\mathbf{r}|n,p_z\rangle = \left(\frac{m\Omega}{\pi\hbar 2^{2n}(n!)^2}\right)^{1/4}\mathcal{H}_n\left(\sqrt{\frac{m\Omega}{\hbar}}r_\perp\right) \times\exp\left[-\frac{m\Omega}{2\hbar}r_\perp^2 + i\frac{p_z}{\hbar}z\right], \tag{32}$$

where $\mathcal{H}_n(x)$ is the nth order Hermite polynomial, we obtain

$$|\langle n',p_z''|\mathbf{r}|n,p_z\rangle|^2 \approx \frac{n\hbar\Omega}{2m\Omega^2}\delta_{n,n'+1}\delta(p_z''-p_z). \tag{33}$$

Inserting this result into Eq. (31), the transition rate from the excited to the ground state by spontaneous decay in our system is found to be

$$A_{01}(p_z',p_z) \approx S(\omega)\delta(p_z''-p_z) = \frac{2e^2\omega^2}{3mc^3}\delta(p_z''-p_z). \tag{34}$$

This result is a modified decay rate of the spontaneous decay rate without any longitudinal recoil, so that the result includes the longitudinal recoil by its momentum, $\hbar k_\parallel$.

In this case, the resonance conditions including the recoil is

$$\hbar k_\parallel = \pm\hbar k_0 = p_z' - p_z, \tag{35a}$$

$$\hbar\omega_R = \hbar\Omega + \frac{1}{2m}(p_z'^2 - p_z^2) \approx \hbar\Omega + \hbar k_\parallel\frac{p_z}{m}, \tag{35b}$$

where $k_\parallel$ has two values, $\pm k_0$, because both types of transition are required to get a low longitudinal temperature. Thus, the spontaneous decay rate is modified as

$$\overline{A}_{01}(p_z',p_z) \approx S(\omega_R)\delta(p_z'-p_z-\hbar k_\parallel) = \frac{2e^2\omega_R^2}{3mc^3}\delta(p_z'-p_z-\hbar k_\parallel). \tag{36a}$$

In order to keep the thermal equilibrium at a finite temperature, another transition rate, called the induced transition rate by the electromagnetic radiation, is required in the equilibrium. In general, the two transitions for the two states are satisfied by the rule of Einstein's A and B coefficients. With only the two transition rates, we can get relations as

$$\tilde{D}_0(p_z',p_z) \approx \frac{\overline{A}_{01}(p_z',p_z)}{1-\exp(-\beta\hbar\omega_R)}, \tag{37}$$

$$\tilde{W}_0(p_z',p_z) = \tilde{D}_0(p_z',p_z)\exp(-\beta\hbar\omega_R), \tag{38}$$

where β is defined as

$$\beta = \frac{1}{k_B T_h}, \tag{39}$$

from the temperature of the heat bath, T_h. However, the interaction of electrons with the microwave causes two-way transitions between the ground and the first excited state. These transitions are known as stimulated transitions, the rates of which are the same. Moreover, electrons of the first excited state radiate photons spontaneously before they reach an equilibrium, and then drop down to the ground state. This kind of radiation is formulated as the cyclotron radiation. Therefore, the total transition rate from the first excited state to the ground state is the sum of the spontaneous decay, the induced emission by the electromagnetic radiation, and the stimulated transition rate by the external microwave, while the transition rate from the ground to the first excited state is the sum of the induced absorption by the electromagnetic radiation and the stimulated transition rate by the external microwave. The time-dependent microwave field, generated in a cavity, can be expressed as

$$E(t) = E_0 \exp[-i(\omega_0 + \delta\omega)t - \omega_0 t/2Q], \tag{40}$$

where Q is defined as 2π times the time-averaged energy stored in the cavity to the energy loss per cycle. The above field is the solution which gives a constant Q. This damped oscillation, as in Eq. (40), does not have a pure frequency, but a superposition of frequencies around $\omega = \omega_0 + \delta\omega$. The Fourier transformation of Eq. (40) leads to a frequency distribution of the energy in the cavity. The result of a Fourier transformation such as

$$\frac{|E(\omega)|^2}{8\pi} = \frac{E_0^2/(4\pi)^2}{(\omega_0/2Q)^2 + (\omega - \omega_0 - \delta\omega)^2} \tag{41}$$

represents energy density as a function of frequency. The microwave-stimulated transition rate is the same as the energy density per unit of emitted photon energy, and the resonance frequency, $\omega_R = \omega_0 + \delta\omega$, is decided by the sum of the transverse energy change and the longitudinal energy change.

As mentioned, our system is simplified as consisting of two levels separated by the energy $\hbar\Omega$. This is acted upon by a microwave field of frequency ω, traveling along the magnetic field line. When an electron absorbs the energy $\hbar\omega$, it must also compensate for the loss of momentum $\hbar k_\parallel$ by the field. The resonance energy is mainly the transition energy between two transverse energy levels, and is corrected by a small longitudinal energy change.

Thus resonance conditions of the microwave are the same as those during spontaneous transition. This implies that ω_0 and $\delta\omega$ are regarded as Ω and $k_\parallel p_z/m$, respectively. The density of states in the microwave field is the normalization of the energy density generated by the microwave field. So the normalized density of states is

$$g(\omega) = \frac{1}{\pi}\frac{\Omega/2Q}{(\Omega/2Q)^2 + (\omega - \Omega - k_\parallel p_z/m)^2}. \tag{42}$$

The microwave-stimulated transition rate can be obtained from the Fermi golden rule, which is used to obtain the spontaneous emission in Eq. (22). For the stimulated transition, the density of states generated by the microwave field should be replaced by the density of states for the spontaneous emission expressed in Eq. (26). In this case, ω_0 is replaced by Ω, and $\Delta\omega$ can be interpreted as $k_\parallel p_z/m$, by the resonance condition. Also, the emission matrix for the stimulated transition is changed to

$$H_{n,n'} = \frac{eE_0}{\sqrt{8\pi}}\langle n', p_z'|\vec{r}_\perp|n, p_z\rangle, \tag{43}$$

where $\vec{r}_\perp$ is perpendicular to the wave vector of the field. Following the same procedure to calculate the spontaneous transition rate, and introducing a microwave intensity as a combination of a dipole momentum and a microwave field strength, the stimulated transition rate by microwave can be expressed as

$$w_s(p_z', p_z) = \frac{I\sigma_0}{\hbar\Omega}\frac{(\gamma/2)^2}{(\gamma/2)^2 + \Delta^2(\omega, k_\parallel, p_z, p_z')}, \tag{44}$$

where the function Δ is defined as $\Delta \equiv \omega - \omega_R(k_\parallel, p_z, p_z')$. The $I\sigma_0$ in w_s is the power of the microwave defined as

$$I\sigma_0 = \frac{e^2E_0^2}{4\pi\hbar\gamma}, \tag{45}$$

and the γ can be determined as Ω/Q from the Q factor of the microwave cavity. Including the resonance condition in the momentum space, the transition rate can be expressed as

$$\widetilde{w}_s(p_z', p_z) = w_s(p_z', p_z)\,\delta(p_z' - p_z - \hbar k_\parallel). \tag{46}$$

Combining the two transition rates, the total emission and the absorption transition rates are changed to

$$\widetilde{D}(p_z', p_z) = \widetilde{D}_0(p_z', p_z) + \widetilde{w}_s(p_z', p_z), \tag{47a}$$

$$\widetilde{W}(p_z', p_z) = \widetilde{W}_0(p_z', p_z) + \widetilde{w}_s(p_z', p_z). \tag{47b}$$

The above two transition rates have their own momentum resonance conditions in themselves, so that the energy resonance condition automatically can be applied to the these two equations. In order to cool the electron plasma properly, we need to use both ways of cooling, which means that we have to apply two microwaves symmetrically in momentum space. The electron that has a positive momentum will lose its longitudinal momentum by the negative longitudinal wave vector $-\hbar k_0$, where k_0 is positively defined as $\hbar\Omega/c$. On the other hand, the electron that has negative momentum will gain longitudinal momentum by the positive longitudinal wave vector $\hbar k_0$. Under these two simultaneous processes, the electrons will lose their longitudinal energy.

We define the following functions for our convenience:

$$D_0(p_z, p_z') = \frac{S(\omega_R)/2}{1 - \exp(-\beta\hbar\omega_R)}, \tag{48a}$$

$$W_0(p_z, p_z') = D_0(p_z, p_z')\exp(-\beta\hbar\omega_R), \tag{48b}$$

where the factor 1/2 appears in Eq. (48a), because a normalized two-way plane wave should be considered a longitudinal wave instead of a one-way plane wave. Now the longitudinal distribution equations of Eq. (19a) and Eq. (19b) can be reduced to

$$\begin{aligned}\frac{\partial f_0(p_z)}{\partial t} = \sum_{k_\parallel=\pm k_0} & [f_1(p_z+\hbar k_\parallel)D_0(p_z+\hbar k_\parallel,p_z) \\ & - f_0(p_z)W_0(p_z,p_z+\hbar k_\parallel)] \\ & + \sum_{k_\parallel=\pm k_0}[f_1(p_z+\hbar k_\parallel)w_s(p_z+\hbar k_\parallel,p_z) \\ & - f_0(p_z)w_s(p_z,p_z+\hbar k_\parallel)],\end{aligned} \quad (49a)$$

$$\begin{aligned}\frac{\partial f_1(p_z')}{\partial t} = \sum_{k_\parallel=\pm k_0} & [f_0(p_z'-\hbar k_\parallel)W_0(p_z'-\hbar k_\parallel,p_z') \\ & - f_1(p_z')D_0(p_z',p_z'-\hbar k_\parallel)] + \sum_{k_\parallel=\pm k_0}[f_0(p_z'-\hbar k_\parallel)w_s(p_z' \\ & -\hbar k_\parallel,p_z') - f_1(p_z')w_s(p_z',p_z'-\hbar k_\parallel)],\end{aligned} \quad (49b)$$

where the ω_R can be determined by $k_\parallel$.

IV. FOKKER-PLANCK EQUATION

In order to calculate the cooling rate of the longitudinal temperature, it is necessary for us to derive a Fokker-Planck equation induced by a microwave and its dissipative and diffusional coefficients. In our analysis of the two level-transition equations, we have supposed that the transition time scale is much smaller than the collision time scale. Without an external microwave field, the collision drives the plasma to a state of equilibrium, but a strong magnetic field causes a shorter spontaneous-decay time scale ($\tau_{sp}^{-1} \approx 2e^2\Omega^2/3mc^3$) than that caused by the interparticle collision alone, because the time scale of the collision between electrons is exponentially smaller by its exponents $-\Omega/\omega_p$ and $-\Omega/\omega_\parallel$ [21,22]. With a comparable microwave intensity applied to the plasma, the plasma profile will evolve in a relatively short time, independently of the relaxation process by the collision. As the equation that is driven by the collision is usually reduced to a Fokker-Planck equation in a long time scale, the shorter time scale transition equations of our strongly magnetized, microwave-stimulated system can be reduced to another Fokker-Planck equation in a relatively short time scale.

In our system, the momentum change, $\hbar k_0 = \hbar\Omega/c$, is much smaller than the standard deviation of the longitudinal distribution, $\langle p_z^2\rangle^{1/2}$. Also, the peak position of the microwave spectrum in momentum space, $m(\Omega-\omega)/k_0$, is much larger than $\hbar k_0$. Since the total longitudinal distribution, $f(p_z) = f_0(p_z)+f_1(p_z)$, is a smooth and slowly-varying function about p_z, we easily find a condition for the distribution,

$$\hbar k_0 \frac{\partial f(p_z)}{\partial p_z} \ll f(p_z). \quad (50)$$

This condition implies that the deforms of the longitudinal distributions by small changes in their longitudinal momenta are very small compared to the original longitudinal distributions. As results of this condition, the two transition equations in their main orders are completely opposite, and the state of the plasma that is represented as the total longitudinal distribution evolves to low longitudinal temperature range in a long time scale. This condition also leads a special relation,

$$\frac{f_1(p_z)}{f_0(p_z)} = \frac{W(p_z,p_z+\hbar k_0)+W(p_z,p_z-\hbar k_0)}{D(p_z,p_z-\hbar k_0)+D(p_z,p_z+\hbar k_0)}, \quad (51)$$

in a short time scale, the spontaneous time scale, to the two distributions, $f_0(p_z)$ and $f_1(p_z)$.

By taking a Taylor expansion of the two level equations, our original transition equations can be written as

$$\begin{aligned}\frac{\partial f_0(p_z)}{\partial t} \approx\; & f_1(p_z)[D(p_z;-\hbar k_0)+D(p_z;\hbar k_0)] \\ & - f_0(p_z)[W(p_z;\hbar k_0)+W(p_z;-\hbar k_0)] \\ & + \hbar k_0 \frac{\partial}{\partial p_z}[f_1(p_z)\{D(p_z;-\hbar k_0)-D(p_z;\hbar k_0)\}] \\ & + \frac{\hbar^2 k_0^2}{2}\frac{\partial^2}{\partial p_z^2}[f_1(p_z)\{D(p_z;-\hbar k_0)+D(p_z;\hbar k_0)\}],\end{aligned} \quad (52a)$$

$$\begin{aligned}\frac{\partial f_1(p_z)}{\partial t} \approx\; & f_0(p_z)[W(p_z;\hbar k_0)+W(p_z;-\hbar k_0)] \\ & - f_1(p_z)[D(p_z;-\hbar k_0)+D(p_z;\hbar k_0)] \\ & + \hbar k_0 \frac{\partial}{\partial p_z}[f_0(p_z)\{W(p_z;\hbar k_0)-W(p_z;-\hbar k_0)\}] \\ & + \frac{\hbar^2 k_0^2}{2}\frac{\partial^2}{\partial p_z^2}[f_0(p_z)\{W(p_z;\hbar k_0)+W(p_z;-\hbar k_0)\}]\end{aligned} \quad (52b)$$

to the order of $\hbar^2 k_0^2$, where the second argument in the parenthesis of D, W, and w_s represents the gain of momentum. This implies that $D_0(p_z;\pm\hbar k_0)$, $W_0(p_z;\pm\hbar k_0)$, and $w_s(p_z;\pm\hbar k_0)$ become

$$D_0(p_z;\pm\hbar k_0) = \frac{\dfrac{e^2\left(\Omega \pm \dfrac{k_0}{m}p_z\right)^2}{3mc^3}}{1-\exp[-\beta(\Omega \pm k_0 p_z/m)]}, \quad (53a)$$

$$W_0(p_z;\pm\hbar k_0) = \frac{\dfrac{e^2\left(\Omega \pm \dfrac{k_0}{m}p_z\right)^2}{3mc^3}\exp[-\beta(\Omega \pm k_0 p_z/m)]}{1-\exp[-\beta(\Omega \pm k_0 p_z/m)]}, \quad (53b)$$

$$w_s(p_z;\pm\hbar k_0) = \frac{I\sigma_0}{\hbar\Omega}\frac{(\gamma/2)^2}{(\gamma/2)^2+(\Delta\omega \pm k_0 p_z/m)^2}, \quad (53c)$$

where the detuning from resonance frequency, $\Delta\omega$ is defined as

$$\Delta\omega \equiv \Omega - \omega. \quad (54)$$

For our convenience, we define the following functions as

$$w_s^T(p_z) = w_s(p_z;-\hbar k_0) + w_s(p_z;\hbar k_0), \tag{55a}$$

$$\Delta w_s(p_z) = w_s(p_z;-\hbar k_0) - w_s(p_z;\hbar k_0), \tag{55b}$$

$$D_0^T(p_z) = D_0(p_z;-\hbar k_0) + D_0(p_z;\hbar k_0), \tag{55c}$$

$$\Delta D_0(p_z) = D_0(p_z;-\hbar k_0) + D_0(p_z;\hbar k_0), \tag{55d}$$

$$W_0^T(p_z) = W_0(p_z;-\hbar k_0) + W_0(p_z;\hbar k_0), \tag{55e}$$

$$\Delta W_0(p_z) = W_0(p_z;-\hbar k_0) - W_0(p_z;\hbar k_0). \tag{55f}$$

By defining the following four functions as combinations of the above six defined functions,

$$D^T(p_z) = D_0^T(p_z) + w_s^T(p_z), \tag{56a}$$

$$W^T(p_z) = W_0^T(p_z) + w_s^T(p_z), \tag{56b}$$

$$\Delta D(p_z) = \Delta D_0(p_z) + \Delta w_s(p_z), \tag{56c}$$

$$\Delta W(p_z) = \Delta W_0(p_z) + \Delta w_s(p_z), \tag{56d}$$

the two transition equations can be rewritten as

$$\frac{\partial f_0(p_z)}{\partial t} \approx f_1(p_z)D^T(p_z) - f_0(p_z)W^T(p_z) + \hbar k_0 \frac{\partial}{\partial p_z}\{f_1(p_z)\Delta D(p_z)\} + \frac{\hbar^2 k_0^2}{2}\frac{\partial^2}{\partial p_z^2}\{f_1(p_z)D^T(p_z)\}, \tag{57a}$$

$$\frac{\partial f_1(p_z')}{\partial t} \approx f_0(p_z)W^T(p_z) - f_1(p_z)D^T(p_z) + \hbar k_0 \frac{\partial}{\partial p_z}\{f_0(p_z)\Delta W(p_z)\} + \frac{\hbar^2 k_0^2}{2}\frac{\partial^2}{\partial p_z^2}\{f_0(p_z)W^T(p_z)\}. \tag{57b}$$

By adding the two transition equations, a partial differential equation for the distribution $f(p_z)$ can be derived as

$$\frac{\partial f(p_z)}{\partial t} \approx -\hbar k_0 \frac{\partial}{\partial p_z}[f_0(p_z)\Delta W(p_z) - f_1(p_z)\Delta D(p_z)] + \frac{\hbar^2 k_0^2}{2}\frac{\partial^2}{\partial p_z^2}[f_0(p_z)W^T(p_z) + f_1(p_z)D^T(p_z)] \tag{58}$$

to the order of $\hbar^2 k_0^2$. From the above equation, the main change in the plasma profile is proportional to $\hbar k_0$. However, the right-hand side of the equation is not an appropriate form for a Fokker-Planck equation. In order to find a Fokker-Planck equation for only $f(p_z)$, the right-hand side of Eq. (58) has to be an expression of $f(p_z)$ rather than of $f_n(p_z)$.

As mentioned before, the condition in Eq. (50) leads a special relation, Eq. (51), between the two state in a short time scale, the spontaneous time scale. After the time scale, the two states can be expressed as functions of the total state, $f(p_z)$, because of a definition, $f(p_z) = f_0(p_z) + f_1(p_z)$. This means that Eq. (58) can be a Fokker-Planck equation in the short time scale. To do that properly, we use Eq. (57a) and Eq. (57b). As an immediate result of combining Eq. (57a) and Eq. (57b),

$$\begin{aligned}\frac{\partial}{\partial t}[W^T(p_z)f_0(p_z) - D^T(p_z)f_1(p_z)] &\approx -[D^T(p_z) + W^T(p_z)][W^T(p_z)f_0(p_z) - D^T(p_z)f_1(p_z)] \\ &\quad + \hbar k_0 \left[D^T(p_z)\frac{\partial}{\partial p_z}\{\Delta W(p_z)f_0(p_z)\} + W^T(p_z)\frac{\partial}{\partial p_z}\{\Delta D(p_z)f_1(p_z)\}\right]\end{aligned} \tag{59}$$

is obtained, to the order of $\hbar k_0$. Equation (59) shows that the function $W^T(p_z)f_0(p_z) - D^T(p_z)f_1(p_z)$ decreases as much as its exponent, $D^T(p_z) + W^T(p_z)$, makes it decrease. Therefore, the function with a large exponent vanishes so rapidly that the stationarity of the function gives a relation between $f_0(p_z)$ and $f_1(p_z)$. The two states, as functions of the total state, can be obtained by eliminating the time derivative term of the equation. With a definition, $f(p_z) = f_0(p_z) + f_1(p_z)$, the two states become

$$\begin{aligned}f_0(p_z) &= \frac{D^T(p_z)}{D^T(p_z) + W^T(p_z)}f(p_z) + \hbar k_0 \frac{D^T(p_z)}{\{D^T(p_z) + W^T(p_z)\}^2}\frac{\partial}{\partial p_z}\left[\Delta W(p_z)\frac{D^T(p_z)}{D^T(p_z) + W^T(p_z)}f(p_z)\right] \\ &\quad + \hbar k_0 \frac{W^T(p_z)}{\{D^T(p_z) + W^T(p_z)\}^2}\frac{\partial}{\partial p_z}\left[\Delta D(p_z)\frac{W^T(p_z)}{D^T(p_z) + W^T(p_z)}f(p_z)\right],\end{aligned} \tag{60a}$$

$$\begin{aligned}f_1(p_z) &= \frac{W^T(p_z)}{D^T(p_z) + W^T(p_z)}f(p_z) - \hbar k_0 \frac{D^T(p_z)}{\{D^T(p_z) + W^T(p_z)\}^2}\frac{\partial}{\partial p_z}\left[\Delta W(p_z)\frac{D^T(p_z)}{D^T(p_z) + W^T(p_z)}f(p_z)\right] \\ &\quad - \hbar k_0 \frac{W^T(p_z)}{\{D^T(p_z) + W^T(p_z)\}^2}\frac{\partial}{\partial p_z}\left[\Delta D(p_z)\frac{W^T(p_z)}{D^T(p_z) + W^T(p_z)}f(p_z)\right]\end{aligned} \tag{60b}$$

to the order of $\hbar k_0$. Inserting these two expressions into Eq. (58) gives the proper form of the Fokker-Planck equation,

$$\frac{\partial f(p_z)}{\partial t} = \frac{\partial}{\partial p_z}\left[K(p_z)f(p_z) + H(p_z)\frac{\partial}{\partial p_z}f(p_z)\right]. \qquad (61)$$

The two coefficients are defined as

$$K(p_z) = \hbar k_0 K_1(p_z) + \hbar^2 k_0^2 K_2(p_z), \qquad (62a)$$

$$H(p_z) = \hbar^2 k_0^2 H_2(p_z). \qquad (62b)$$

In order to obtain the coefficients $K_1(p_z)$, $K_2(p_z)$, and $H_2(p_z)$ properly, we need to find the main order of the components in the coefficients. The coefficients have $D_0^T(p_z)$, $W_0^T(p_z)$, $\Delta D_0(p_z)$, $\Delta W_0(p_z)$, $w_s^T(p_z)$, and $\Delta w_s(p_z)$ as their components. The coefficients are the functions of the combinations of the six components. Since $w_s^T(p_z)$, and $\Delta w_s(p_z)$ are very localized at $p_z = \mp m\Delta\omega/k_0$, and the other four components are slow-varying functions in p_z space, we know easily that the first nonvanishing terms of the four components, $D_0^T(p_z)$, $W_0^T(p_z)$, $w_s^T(p_z)$, and $\Delta w_s(p_z)$, do not depend on $\hbar k_0$. The main non-vanishing terms of the remaining two components, $\Delta D_0(p_z)$ and $\Delta W_0(p_z)$, are different from those of the other four components. By expanding $\Delta W_0(p_z)$ and $\Delta D_0(p_z)$ to the order of $\hbar k_0$,

$$\Delta W_0(p_z) = -\frac{2}{\hbar\Omega}\frac{\hbar k_0}{m}\frac{S_0\exp(-\beta\hbar\Omega)}{1-\exp(-\beta\hbar\Omega)}p_z + \beta\frac{\hbar k_0}{m}\frac{S_0}{\{1-\exp(-\beta\hbar\Omega)\}^2}p_z, \qquad (63a)$$

$$\Delta D_0(p_z) = \Delta W_0(p_z)\exp(-\beta\hbar\Omega) - \frac{\hbar k_0}{m}\frac{S_0}{1-\exp(-\beta\hbar\Omega)}, \qquad (63b)$$

we know that the nonvanishing terms in both components proportional to $\hbar k_0$.

Recalling the main orders of $\Delta W_0(p_z)$ and $\Delta D_0(p_z)$, we can obtain $K_1(p_z)$, $K_2(p_z)$, and $H_2(p_z)$. The results are

$$K_1(p_z) = -\frac{[D_0^T(p_z) - W_0^T(p_z)]\Delta w_s(p_z)}{D^T(p_z) + W^T(p_z)}, \qquad (64a)$$

$$\begin{aligned} K_2(p_z) = &-\frac{1}{\hbar k_0}\frac{D^T(p_z)\Delta W_0(p_z) - W^T(p_z)\Delta D_0(p_z)}{D^T(p_z)+W^T(p_z)} \\ &-\frac{2\Delta w_s(p_z)}{\{D^T(p_z)+W^T(p_z)\}^2}\left[D^T(p_z)\frac{\partial}{\partial p_z}\left\{\frac{D^T(p_z)}{D^T(p_z)+W^T(p_z)}\Delta w_s(p_z)\right\}W^T(p_z)\frac{\partial}{\partial p_z}\left\{\frac{W^T(p_z)}{D^T(p_z)+W^T(p_z)}\Delta w_s(p_z)\right\}\right] \\ &-\frac{\partial}{\partial p_z}\frac{D^T(p_z)W^T(p_z)}{D^T(p_z)+W^T(p_z)}, \end{aligned} \qquad (64b)$$

$$H_2(p_z) = -\frac{2\Delta w_s^2(p_z)}{\{D^T(p_z)+W^T(p_z)\}^2}\frac{\{D^T(p_z)\}^2+\{W^T(p_z)\}^2}{D^T(p_z)+W^T(p_z)} + \frac{D^T(p_z)W^T(p_z)}{D^T(p_z)+W^T(p_z)}. \qquad (64c)$$

As mentioned before, the main change of the plasma profile is proportional to $\hbar k_0$. The dissipation of the plasma is more dominant than the diffusion because the main ratio of the two coefficients is in the order of $\hbar k_0$.

V. STEADY STATES

Let us apply the Fokker-Planck equation of Eq. (61) with $K_1(p_z)$, $K_2(p_z)$, and $H_2(p_z)$, in Eq. (64a), Eq. (64b), Eq. (64c). We may consider two extreme cases: the case in which no microwave is applied, and the case in which there is only microwave-stimulated transition.

In the first case, where there is no microwave, the state is equilibrated by both spontaneous decay and induced transition. Here, the induced transition is generated by the electromagnetic radiation. Whenever an electron in plasma experiences a transition by both the spontaneous and the induced transition, the electron loses its longitudinal energy by longitudinal momentum loss, $\hbar k_0$, repeating the process until the plasma reaches an equilibrium. As a result of this equilibration process, the stationary temperature of the plasma should be exactly the same as the heat bath temperature of the plasma. Mathematically, this process can be described by the classical Fokker-Planck equation, which has a constant diffusion coefficient and a dissipative coefficient which is proportional to p_z. Therefore, the Fokker-Planck equation of Eq. (61) should be reduced to a classical Fokker-Planck equation. Applying $w_s(p_z)=0$ to $K_1(p_z)$, $K_2(p_z)$, and $H_2(p_z)$, with $\Delta W_0(p_z)$, and $\Delta D_0(p_z)$, in Eq. (63a) and Eq. (63b), we can obtain $K_2(p_z)$ and $H_2(p_z)$, as

$$K_2(p_z) = \frac{\beta}{2m}\frac{S_0}{\sinh(\beta\hbar\Omega)}p_z, \qquad (65a)$$

$$H_2(p_z) = \frac{1}{2}\frac{S_0}{\sinh(\beta\hbar\Omega)}, \tag{65b}$$

while $K_1(p_z)$ vanishes. Since $K_2(p_z)$ and $H_2(p_z)$ are expressed as $K_2 p_z$, and H_2, the dissipative time scale and the stationary temperature can be calculated from the well-known results of the classical Fokker-Planck equation:

$$\frac{1}{\tau_e} = 2\hbar^2 k_0^2 K_2, \tag{66a}$$

$$k_B T_\infty = \frac{1}{m}\frac{\hbar^2 k_0^2 H_2}{\hbar^2 k_0^2 K_2}, \tag{66b}$$

where τ_e is the exponent decay time, and T_∞ is the stationary temperature as time goes infinity. A direct calculation gives

$$\frac{1}{\tau_e} = \beta\frac{\hbar^2 k_0^2}{2m}\frac{2S_0}{\sinh(\beta\hbar\Omega)}, \tag{67a}$$

$$T_\infty = T_h. \tag{67b}$$

Note that the Fokker-Planck equation with $K_2 p_z$ and H_2 as $K_2(p_z)$ and $H_2(p_z)$ leads to a Gaussian profile as the stationary state, which means that the distribution is unchanged when we take T_h as the initial temperature. The dissipative exponent, $\nu_e \equiv 1/\tau_e$, is proportional to $\hbar^2 k_0^2$, while the exponent with the microwave is generally proportional to $\hbar k_0$. Therefore, microwave cooling is $\hbar k_0$ times faster than equilibration by both the spontaneous and the induced transition.

The second case, of only microwave-stimulated transition, produces almost the same result as the case of no microwave. In this case, the transition coefficients, $D^T(p_z)$ and $W^T(p_z)$, are the same as $w_s^T(p_z)$, so that the two coefficients of the Fokker-Planck equation are reduced to

$$K(p_z) = \frac{\hbar^2 k_0^2}{2}\left[\frac{\partial}{\partial p_z} w_s^T(p_z) - \frac{\Delta w_s(p_z)}{w_s^T(p_z)}\frac{\partial}{\partial p_z}\Delta w_s(p_z)\right], \tag{68a}$$

$$H(p_z) = \frac{\hbar^2 k_0^2}{2}\left[w_s^T(p_z) - \frac{\Delta w_s^2(p_z)}{w_s^T(p_z)}\right]. \tag{68b}$$

The main difference between these coefficients and those in the case of both spontaneous and stimulated transition is that the dissipative coefficient, $K(p_z)$, is mainly proportional to $\hbar^2 k_0^2$ rather than to $\hbar k_0$. This implies that cooling with only microwave-stimulated transition is so slow there is effectively no cooling within the desired short time scale. This phenomenon can be understood as follows. Without the spontaneous decay and the induced transition by the electromagnetic radiation, an electron in the ground state absorbs a microwave photon and then moves up one in its Landau state. At the same time, the electron loses its longitudinal momentum, $\hbar k_0$, so that it also loses its longitudinal energy. By contrast, an electron in the excited state emits a photon of the same energy as it would in the case of absorption by a microwave. As a result of the emission, the electron moves down to the ground state and gains longitudinal momentum, $\hbar k_0$. Since there is no spontaneous decay, but only the same amount of microwave in both the ground state and the first excited state, the number of electrons in the two states is almost the same. As shown in Eq. (59) and explained before, the time scale needed to reach the situation in which the number of electrons is almost the same is very short, compared to the cooling time scale. As the stationary solutions of Eq. (59), the two states are

$$f_0(p_z) \approx \frac{1}{2}f(p_z) + \frac{\hbar k_0}{4w_s^T(p_z)}\frac{\partial}{\partial p_z}[\Delta w_s(p_z)f(p_z)], \tag{69a}$$

$$f_1(p_z) \approx \frac{1}{2}f(p_z) - \frac{\hbar k_0}{4w_s^T(p_z)}\frac{\partial}{\partial p_z}[\Delta w_s(p_z)f(p_z)]. \tag{69b}$$

Obviously, the difference between the two states is proportional to $\hbar k_0$, which means that the two states are almost identical. Because the number of electrons that lose their longitudinal momenta is exactly the same as the number that gain longitudinal momenta, there is no more longitudinal cooling in this case. The spontaneous terms mainly prevent the numbers of electrons in the two states from being the same, so that longitudinal cooling can continue.

We have investigated cooling without microwave-stimulated transition and cooling with only microwave-stimulated transition. Without the stimulated transition, the Fokker-Planck equation reduces to exactly what we want: an equation that always gives a Gaussian profile in time, when we take a Gaussian profile as its initial longitudinal distribution; a stationary temperature that is exactly the same as the heat bath temperature; and a dissipative exponent proportional to $\hbar^2 k_0^2$. The Fokker-Planck equation in the case of only microwave-stimulated transition leads to a rapid transition in Landau level, which leads to the same populations in the two Landau levels very quickly. However, the dissipative term in the equations is proportional to $\hbar^2 k_0^2$. In both cases, the cooling rate that is determined by the dissipative term is proportional to $\hbar^2 k_0^2$. On the other hand, the cooling rate with both transitions is mainly proportional to $\hbar k_0$. This implies that the microwave-stimulation makes the cooling rate $\hbar k_0$ times faster than the cooling rate without the stimulation. Therefore, longitudinal cooling by microwave-stimulation can be faster than natural cooling brought about by changing the heat bath temperature.

VI. TEMPERATURE RELAXATION

We have derived a Fokker-Planck equation induced by a microwave and its dissipative and diffusion coefficients, and showed that the deformed equations for two extreme cases are well explained physically. With the dissipative and the diffusion coefficients of the Fokker-Planck equation, the cooling rate of the longitudinal temperature can be calculated. In order to get the relaxation equation for the longitudinal temperature, we will deform the Fokker-Planck equation as a form of dimensionless equation.

First of all, we need to introduce a dimensionless time to the system. The particle number ratio between the ground and the first excited state depends on the spontaneous and the stimulated transition rates. More rigorously, the ratio depends on the relative rate of the stimulated transition rate to the spontaneous transition rate. Unlike spontaneous transitions, which occur with every electron, microwave-stimulated transitions occur only with electrons with velocities that satisfy a resonance condition. This means that the spontaneous transition is more dominant than the stimulated transition, except for some ranges in velocity space where the localized stimulated transition is dominant.

For this reason we define a dimensionless time as

$$\tau = S_0 t, \tag{70}$$

where $S_0 = 2e^2\Omega^2/3mc^3$ is the spontaneous decay rate. As units of momentum and microwave intensity, we employ $p_T = (mk_B T_h)^{1/2}$ and $E_s = (I\sigma_0/\hbar\Omega)/S_0$, where T_h is the heat bath temperature. With the two units, the microwave transition rate can be dimensionless. The rate is then

$$\overline{w_s}(q_z;\pm\hbar k_0/p_T) = E_s \frac{\left(\dfrac{m\gamma}{2k_\| p_T}\right)^2}{\left(\dfrac{m\gamma}{2k_\| p_T}\right)^2 + \left(\dfrac{m\Delta\omega}{k_\| p_T} \mp q_z\right)^2}, \tag{71}$$

and the Fokker-Planck equation is

$$\frac{\partial f(q_z)}{\partial \tau} = \frac{\partial}{\partial q_z}\left[\overline{K}(q_z) f(q_z) + \overline{H}(q_z)\frac{\partial}{\partial q_z} f(q_z)\right], \tag{72}$$

where $q_z = p_z/p_T$ is the dimensionless momentum, and the $f(q_z)$ is the dimensionless distribution. The $\overline{K}(q_z)$ and $\overline{H}(q_z)$ are defined from Eq. (64a), Eq. (64b) and Eq. (64c) by redefining the four functions as

$$\overline{D^T}(q_z) = \frac{D^T(p_z)}{S_0}, \tag{73a}$$

$$\overline{W^T}(q_z) = \frac{W^T(p_z)}{S_0}, \tag{73b}$$

$$\overline{w_s^T}(q_z) = \frac{w_s(p_z)}{S_0}, \tag{73c}$$

$$\Delta\overline{w_s}(q_z) = \frac{\Delta w_s(p_z)}{S_0}, \tag{73d}$$

and by replacing $\hbar k_0$ to $\hbar k_0/p_T$.

In order to solve Eq. (72) numerically, we will use Galerkin FEM method [28]. By applying the method, Eq. (72) can be expressed in discrete regime and the values of $f(q_z)$ are calculated on a regular set of grid points l, with appropriate q_z. To get intermediated values, a set of local weight functions around the grid points can be applied with some restrictive conditions. Then the continuous function $f(q_z)$ is approximated by the sum of a linear combination of the local weighting functions as a form of

$$f(q_z) = \sum_l F^l(t)\alpha_l(q_z), \tag{74}$$

which can be interpreted as the distribution for finite-size momentum, sampled on the grids through an appropriate interpolation. In this case, the first order weighting is enough to achieve the interpolation, so that the local weighting can be defined as

$$\alpha_l(q_z) = \frac{1}{\Delta q_z{}^2}(\Delta q_z - |q_z - q_{z,l}|), \tag{75}$$

where Δq_z is the length between two nearest grid points, and the function vanishes outside the range, $q_{z,l-1} < q_z < q_{z,l+1}$.

By applying Eq. (74) and Eq. (75) for parallel distribution equations, Eq. (72) can be reduced to a finite dimensional matrix equation in the form of

$$\sum_l \frac{\partial}{\partial\tau} F^l(\tau)\overline{A_{lm}} = -\sum_l [F^l(\tau)\overline{K_{lm}} + F^l(\tau)\overline{H_{lm}}], \tag{76}$$

where $F^l(\tau)$ is the value of $f(q_z)$ at $q_z = q_{z,l}$, and the coefficients of equation can be calculated as

$$\overline{A_{lm}} = \int dq_z \alpha_l(q_z)\alpha_m(q_z), \tag{77a}$$

$$\overline{K_{lm}} = \int dq_z \alpha_l(q_z)\overline{K}(q_z)\frac{\partial}{\partial q_z}\alpha_m(q_z), \tag{77b}$$

$$\overline{H_{lm}} = \int dq_z \frac{\partial}{\partial q_z}\alpha_l(q_z)\overline{H}(q_z)\frac{\partial}{\partial q_z}\alpha_m(q_z). \tag{77c}$$

This equation serves to obtain the time evolution of the distribution function. After solving the matrix equation, the distribution can be represented by the coefficients, $F_l(t)$. Similarly, we can find a longitudinal temperature relaxation equation by multiplying q_z^2 in Eq. (72). The final discretized equation is

$$\sum_l \frac{\partial}{\partial\tau} F^l(\tau)\overline{T_l} = -\sum_l [F^l(\tau)\overline{\Lambda_l} + F^l(\tau)\overline{\Sigma_l}], \tag{78}$$

where the coefficients are

$$\overline{T_l} = \int dq_z \alpha_l(q_z) q_z^2, \tag{79a}$$

$$\overline{\Lambda_l} = 2\int dq_z \alpha_l(q_z) q_z \overline{K}(q_z), \tag{79b}$$

$$\overline{\Sigma_l} = 2\int dq_z \frac{\partial}{\partial q_z}\alpha_l(q_z) q_z \overline{H}(q_z). \tag{79c}$$

In Eq. (78), the left-hand side represents the time evolution of the longitudinal temperature, and the first term and the second term of right-hand side represent the dissipative and the diffusive term, respectively. Equation (78) is the equation to get the evolution of the longitudinal temperature numerically.

The exact expression for the evolution of the longitudinal temperature can be obtained from Eq. (72) by multiplying $q_z^2/2$ and then integrating that. The expression is

$$\frac{\dot{T}_{\parallel}(t)/2}{S_0 T_{\parallel}^{(0)}} = -\int dq_z q_z \left[\overline{K}(q_z) f(q_z) + \overline{H}(q_z) \frac{\partial}{\partial q_z} f(q_z) \right], \quad (80)$$

where the time derivative $\dot{T}_{\parallel}(t)$ is not dimensionless but expressed in the unit of sec. As mentioned, the dominant term in the right-hand side is proportional to $\hbar k_0/p_T$ in the dissipative term and the other terms are $\hbar k_0/p_T$ times smaller than the dominant one. So, with only the dominant term the Eq. (80) can be written as

$$\frac{\dot{T}_{\parallel}(t)}{T_{\parallel}^{(0)}} \approx -\nu_e(B, I\sigma_0, T_{\perp}; T_{\parallel}). \quad (81)$$

The right-hand side from the previous equation is defined as

$$\nu_e(B, I\sigma_0, T_{\perp}; T_{\parallel}) = 2S_0 \frac{\hbar k_0}{p_T} \int dq_z q_z \overline{K_1}(q_z) f(q_z), \quad (82)$$

where $\overline{K_1}(q_z)$ is

$$\overline{K_1}(q_z) = -\frac{\overline{D_0^T}(q_z) - \overline{W_0^T}(q_z)}{\overline{D^T}(q_z) + \overline{W^T}(q_z)} \Delta \overline{w_s}(q_z). \quad (83)$$

Now we will discuss results for rescaling parameters. A general rule for changing all parameters cannot be obtained, but we can get a useful relation for the cooling rate for a special case. Since $\overline{D^T}(q_z)$, $\overline{W^T}(q_z)$, and $\overline{w_s}(q_z)$ are invariant for rescaling B and $I\sigma_0/\hbar\Omega$ to κB and $\kappa^2 I\sigma/\hbar\Omega$, respectively, $\overline{K_1}(q_z)$ is also invariant for rescaling. k_0 and S_0 are expanded to κk_0 and $\kappa^2 S_0$, respectively, by rescaling. We must also consider the rescaling of the transverse temperature. The rescaling of B forces Ω to be rescaled to $\kappa\Omega$, so that $T_{\perp}$ should be rescaled to $\kappa T_{\perp}$, to keep the ratio between the ground and the first excited state the same during cooling. This can be done by rescaling only the initial temperature $T_{\perp}^{(0)}$ to $\kappa T_{\perp}^{(0)}$. Therefore, a relation for rescaling,

$$\nu_e(\kappa B, \kappa^2 I\sigma_0, \kappa T_{\perp}; T_{\parallel}) = \kappa^3 \nu_e(B, I\sigma_0, T_{\perp}; T_{\parallel}), \quad (84)$$

can be obtained.

VII. COOLING CONDITIONS OF NON-NEUTRAL PLASMA

To reach the desired cooling limit provided by the two level-transition equations in our theory without any contradiction, some restrictive conditions on the cooling process are necessary. One of these was presented previously. The quantum limit on the transverse motion of strongly magnetized plasma should be effective during the simulation. The transverse temperatures during the simulation should always be lower than the Landau temperature. Also, it is assumed that the transverse temperature is so low that most of the particles are in the ground or the first excited state, which means the number of particles in the second and higher excited states is always small compared to the number in the two lowest states. Therefore, the transverse temperatures are low enough to keep most of the particles in the two lowest states during the simulation.

To cool the particles properly without contradiction to the theory that we have suggested, it is necessary to find a condition related to the frequency and the frequency width of the microwave spectrum applied to the particles. From the previous theoretical treatment, we know the frequency must be slightly smaller than the gyrofrequency, so that the microwave can make a particle move up one in Landau level and simultaneously reduce the longitudinal energy. However, if the frequency width of the microwave spectrum is large enough to exceed a certain limit (which will be investigated in detail later in this section), our theory encounters two serious problems. One is caused by spin resonance frequency, which is automatically larger than the Landau frequency, because $g/2 \approx 1.001$ is slightly larger than unity. If the width is large enough that

$$\frac{\gamma}{2} > \left(\frac{g}{2} - 1 \right) \Omega, \quad (85)$$

where γ is the normal frequency width of the microwave, Ω/Q, then some of the particles will change their spin states from spin-up to spin-down, instead of moving up in Landau level, and this will cause more longitudinal energy change. However, this kind of effect breaks our two level assumption. In order to avoid this problem, the Q factor should be larger than 10^3.

The other problem encountered by our theory is that the cooling is very effective as long as the width in momentum space, $\gamma/k_{\parallel}$, is smaller than the standard deviation of the longitudinal profile in momentum space. When a microwave with a wave number $k_{\parallel}$ and a frequency ω_R, which are defined in Eq. (35a) and Eq. (35b), propagates along the magnetic field in the plasma, an electron with longitudinal momentum p_z along the magnetic field experiences cooling or heating, because the usual Doppler effect, a shifted frequency,

$$\omega' = \omega + k_{\parallel} \frac{p_z}{m}, \quad (86)$$

has interacting terms between the wave vector $k_{\parallel}$ and the longitudinal momentum p_z. If ω' coincides with the electron cyclotron frequency Ω, then resonant absorption of the wave energy by electrons will take place. This phenomenon is effective in the vicinity of the resonance frequency in momentum space, where particles lose their momenta so that they move into lower momentum range. Because of the resonance condition, all particles near the resonance frequency are shifted into lower momentum range. Moreover, if the resonance frequency is well-chosen, then the particles can rapidly lose most of their momenta, and the longitudinal temperature can decrease until the the width of longitudinal distribution is almost the same as the width of the microwave spectrum in momentum space, $\gamma/k_{\parallel}$. After this point, the cooling rate of the longitudinal motion will slow down. This gives us a condition for the microwave applied to the particles: The initial width of the microwave spectrum in momentum space should be less than the standard deviation of the initial longitudinal distribution [29]. The condition in mathematical form,

$$\frac{\delta\omega}{k_{\parallel}} < \frac{\sqrt{\langle p_z^2 \rangle}}{m}, \tag{87}$$

where $\delta\omega = \gamma/2$ is the frequency width of the microwave spectrum, implies that a smaller frequency width of the microwave spectrum gives a lower temperature of the final distribution. However, if the initial frequency width of the microwave is too small, the number of particles involved with the microwave may be so small that the cooling time will be too long. So, in order to find the fastest way to cool the particles, we should start with a larger frequency width. In that case, the initial cooling rate will be higher, but the rate will slow down sooner, so that it will again take too long to reach the critical temperature ($\Gamma_{\parallel} \approx 170$). Therefore, the solution to the problem is to change to a smaller frequency width at the instant that Eq. (87) is broken, in order to allow the electrons to continue to cool.

VIII. ASYMPTOTIC BEHAVIOR OF LONGITUDINAL DISTRIBUTION

As mentioned previously, applying a microwave with Doppler-shifted resonance frequency to electrons causes longitudinal cooling as they lose their longitudinal momenta. The cooling continues until most of electrons escape from the range of the strong microwave in longitudinal momentum space, interacting instead with the weak microwave in the low longitudinal momentum range. In this case, the diffusion coefficient of the Fokker-Planck equation is almost constant in the range, and the dominant term of the dissipative coefficient is proportional to momentum, p_z. These two coefficients leads the longitudinal distribution of the electrons to a Gaussian distribution in the weak microwave range. The two distributions, $f_0(p_z)$ and $f_1(p_z)$, also become Gaussian distributions in the range. So, one distribution automatically has to be decided by the relation

$$\frac{f_1(p_z)}{f_0(p_z)} \approx n_{01}, \tag{88}$$

where n_{01} is the ratio between the ground and first excited state, N_1/N_0.

The relation in Eq. (88) with Eq. (58) gives another Fokker-Planck equation in weak microwave range, which has a different form from the Fokker-Planck equation that we derived previously. Both Fokker-Planck equations have two coefficients, dissipative and diffusion coefficients. The two coefficients are represented as functions of the intensity of the microwave, the microwave cavity factor and the relative detuning frequency from the gyrofrequency, $\Delta\omega$ ($\Delta\omega \equiv \Omega - \omega$). The intensity of the microwave that is required to keep a constant ratio, n_{01}, can be obtained from the comparison of two dissipative coefficients of those two Fokker-Planck equations in the weak microwave range. The intensity can be represented as a function of the microwave cavity factor, Q, and $\Delta\omega$. Applying the intensity that we obtained to the coefficients of the Fokker-Planck equation in weak microwave range, we can also obtain the stationary temperature and the asymptotic cooling rate as functions of Q and $\Delta\omega$.

For more general analysis in the weak microwave range, we have used dimensionless quantities instead of Q and $\Delta\omega$. The new dimensionless quantities are

$$\frac{1}{\overline{Q}} \equiv \frac{m\gamma}{k_0 p_T}, \tag{89a}$$

$$\overline{P} \equiv \frac{\Delta\omega}{\gamma/2}, \tag{89b}$$

where $1/\overline{Q}$ represents the relative frequency-width of the microwave spectrum to the standard deviation of the initial plasma profile in the dimensionless momentum space. With the aid of these definitions, another parameter as a combination of the above two parameters can be introduced for computational convenience. The new parameter, $\overline{P}/2\overline{Q}$, represents the central position of the microwave spectrum as the unit of standard deviation of the initial plasma profile in the dimensionless momentum space. For the set of $(\overline{Q}, \overline{P}/2\overline{Q})$, the cooling rates and transverse temperatures can be calculated.

In the low momentum range where the intensity of microwave is very weak, $w_s^T(p_z)$ and $\Delta w_s(p_z)$ become

$$w_s^T(p_z) \approx -\frac{I\sigma_0}{\hbar\Omega}\frac{2}{1+\overline{P}^2}, \tag{90a}$$

$$\Delta w_s(p_z) \approx \frac{I\sigma_0}{\hbar\Omega}\frac{8\overline{PQ}}{(1+\overline{P}^2)^2}\frac{p_z}{p_T}, \tag{90b}$$

to the lowest order of p_z. Applying the Eq. (88), to Eq. (58), we finally obtain an asymptotic Fokker-Planck equation:

$$\begin{aligned}\frac{\partial f(p_z)}{\partial t} \approx{}& \hbar k_0 \frac{\partial}{\partial p_z}\left[\frac{1-n_{01}}{1+n_{01}}\Delta w_s(p_z) f(p_z)\right] \\ &+ \frac{\hbar^2 k_0^2}{2}\frac{\partial^2}{\partial p_z^2}\left[\left\{\frac{1}{1+n_{01}}[W_0^T(p_z)+\Delta w_s(p_z)]\right.\right. \\ &\left.\left.+\frac{n_{01}}{1+n_{01}}[D_0^T(p_z)+\Delta w_s(p_z)]\right\} f(p_z)\right].\end{aligned} \tag{91}$$

The first term in the right-hand side of Eq. (91) is the approximation of $K_1(p_z)$, defined in Eq. (64a). The asymptotic Fokker-Planck equation can be rewritten as

$$\frac{\partial f(p_z)}{\partial t} \approx \hbar k_0 \frac{\partial}{\partial p_z}[K_1(p_z) f(p_z)] + \hbar^2 k_0^2 \frac{\partial}{\partial p_z}\left[H_2(p_z)\frac{\partial}{\partial p_z} f(p_z)\right], \tag{92}$$

where $K_1(p_z)$ and $H_2(p_z)$ are determined as

$$K_1(p_z) \approx \frac{1-n_{01}}{1+n_{01}}\frac{I\sigma_0}{\hbar\Omega}\frac{8\overline{PQ}}{(1+\overline{P}^2)^2}\frac{p_z}{p_T}, \tag{93a}$$

$$H_2(p_z) \approx \left(\frac{\exp(-\beta\hbar\Omega)}{1-\exp(-\beta\hbar\Omega)} + 2\frac{I\sigma_0/\hbar\Omega}{S_0}\frac{1}{1+\overline{P^2}} \right)\frac{1}{1+n_{01}}\frac{S_0}{2} + \left(\frac{1}{1-\exp(-\beta\hbar\Omega)} + 2\frac{I\sigma_0/\hbar\Omega}{S_0}\frac{1}{1+\overline{P^2}} \right)\frac{n_{01}}{1+n_{01}}\frac{S_0}{2}, \tag{93b}$$

from Eq. (90a) and Eq. (90b). From the direct expansion of $K_1(p_z)$, defined in Eq. (64a) to the order of p_z, we get

$$K_1(p_z) \approx \frac{1}{\frac{1+\exp(-\beta\hbar\Omega)}{1-\exp(-\beta\hbar\Omega)} + 4\frac{I\sigma_0/\hbar\Omega}{S_0}\frac{1}{1+\overline{P^2}}} \times \frac{I\sigma_0}{\hbar\Omega}\frac{8\overline{PQ}}{(1+\overline{P^2})^2}\frac{p_z}{p_T}, \tag{94}$$

where T_h is heat bath temperature and S_0 is the spontaneous decay rate. From Eq. (93a) and Eq. (94), we obtain the asymptotic microwave intensity for constant ratio, n_{01}. The asymptotic intensity is

$$\frac{I\sigma_0}{\hbar\Omega} \approx S_0\frac{(1+\overline{P^2})}{4}\left[\frac{1+n_{01}}{1-n_{01}} - \frac{1+\exp(-\beta\hbar\Omega)}{1-\exp(-\beta\hbar\Omega)} \right], \tag{95}$$

which shows that the intensity can be decided only by $\overline{P}$.

Equation (92) can be rewritten as

$$\frac{\partial f(p_z)}{\partial t} \approx \hbar k_0 K_1 \frac{\partial}{\partial p_z}[p_z f(p_z)] + \hbar^2 k_0^2 H_2 \frac{\partial^2}{\partial p_z^2} f(p_z), \tag{96}$$

where K_1 and H_2 are

$$K_1 \approx \frac{1-n_{01}}{1+n_{01}}\frac{I\sigma_0}{\hbar\Omega}\frac{8\overline{PQ}}{(1+\overline{P^2})^2}\frac{1}{p_T}, \tag{97a}$$

$$H_2 \approx \left(\frac{\exp(-\beta\hbar\Omega)}{1-\exp(-\beta\hbar\Omega)} + 2\frac{I\sigma_0/\hbar\Omega}{S_0}\frac{1}{1+\overline{P^2}} \right)\frac{1}{1+n_{01}}\frac{S_0}{2} + \left(\frac{1}{1-\exp(-\beta\hbar\Omega)} + 2\frac{I\sigma_0/\hbar\Omega}{S_0}\frac{1}{1+\overline{P^2}} \right)\frac{n_{01}}{1+n_{01}}\frac{S_0}{2}. \tag{97b}$$

The solution of Eq. (96) is well known. If a Gaussian state is the initial state with $T_\parallel^0$ as the initial longitudinal temperature, the state is still Gaussian until it reaches a stationary state. Only the kinetic temperature varies during the evolution. With $T_\parallel^\infty$ as its final temperature, the solution is

$$f(p_z) = \frac{1}{\sqrt{2\pi k_B T_\parallel(t)}}\exp\left[-\frac{p_z^2}{2k_B T_\parallel(t)} \right], \tag{98}$$

where the longitudinal temperature is

$$T_\parallel(t) = T_\parallel^\infty + (T_\parallel^0 - T_\parallel^\infty)e^{-2\hbar k_0 K_1 t}. \tag{99}$$

This shows that $T_\parallel(t) - T_\parallel^\infty$ has $2\hbar k_0 K_1$ as its exponent decay rate. In other words, the exponent for logarithm to base 10 is

$$\nu = -2\hbar k_0 K_1 \log_{10} e. \tag{100}$$

Whatever the initial temperature is, $T_\parallel^\infty$ will be the temperature of stationary state. So, $T_\parallel^\infty$ can be determined as the kinetic temperature of the stationary state. The stationary solution of Eq. (96) implies that the temperature is

$$k_B T_\parallel^\infty = \frac{1}{m}\frac{\hbar^2 k_0^2 H_2}{\hbar k_0 K_1}. \tag{101}$$

Applying the two coefficients, K_1 and H_2, to Eq. (101) with Eq. (95), we obtain the stationary temperature

$$k_B T_\parallel^\infty = \frac{\left[\left(\frac{1+n_{01}}{1-n_{01}} \right)^2 - 1 \right]}{\left[\frac{1+n_{01}}{1-n_{01}} - \frac{1+\exp(-\beta\hbar\Omega)}{1-\exp(-\beta\hbar\Omega)} \right]}\frac{1+\overline{P^2}}{8\overline{P}}\hbar\gamma, \tag{102}$$

which depends on $\overline{P}$, $\overline{Q}$, and n_{01}. The temperature of Eq. (102) agrees qualitatively to Stenholm's result for laser cooling [29]. The cooling temperature is minimized at $\overline{P}=1$ for the constant $\overline{Q}$.

With K_1 of Eq. (97a), the cooling rate of Eq. (100) is

$$\nu = -2\log_{10} e \frac{\hbar k_0}{p_T}\frac{1-n_{01}}{1+n_{01}}\frac{I\sigma_0}{\hbar\Omega}\frac{8\overline{PQ}}{(1+\overline{P^2})^2}. \tag{103}$$

For a constant $\overline{Q}$ and a constant microwave intensity without any consideration of a constant ratio between the two transverse quantum levels, the rate is maximized at $\overline{P}=1/\sqrt{3}$, which is also the same as Stenholm's result [29].

However, our conclusion is that the microwave intensity should be determined by $\overline{P}$ in Eq. (95), which is obtained from the constant ratio between two transverse quantum levels. By combining Eq. (95) and Eq. (103), the asymptotic cooling rate is

$$\nu = -4\log_{10} e S_0 \frac{\hbar k_0}{p_T}\frac{\overline{PQ}}{1+\overline{P^2}}\left[1 - \frac{1-n_{01}}{1+n_{01}}\frac{1+\exp(-\beta\hbar\Omega)}{1-\exp(-\beta\hbar\Omega)} \right]. \tag{104}$$

In the case of the constant ratio and the constant $\overline{Q}$, the maximum cooling occurs at $\overline{P}=1/\sqrt{2}$ instead of $\overline{P}=1/\sqrt{3}$.

However, the process we have applied to find the maximum cooling rate is based on the assumption that the electron plasma profile is far from the strongly peaked microwave in momentum space. On that assumption, we have applied a linear approximation to the system. This means that $\overline{P}/2\overline{Q}$, which represents the central position of the microwave spectrum in units of p_T, is so large that most of the electrons do not interact with the strong microwave. Also, the frequency width of the microwave spectrum, $1/\overline{Q}$, in units of p_T, is so small that the range of the strong microwave is narrow. Since more than 99% of the electrons reside between $-3p_T$ and $3p_T$, $\overline{P}/2\overline{Q}$ should be roughly larger than 4. As a result of this consideration, $1/\overline{Q}$ should be roughly larger than $8\sqrt{2}$, which means that the width of the microwave is initially much larger than p_T, the thermal average momentum of the initial longitudinal distribution. This is not an appropriate result in reality, and the large width of the microwave contradicts Eq. (87) initially. Therefore it seems that maximum cooling cannot be found at $\overline{P}=1/\sqrt{2}$, because $\overline{P}=1/\sqrt{2}$ automatically contradicts our assumption. Therefore, the frequency width of the microwave spectrum should be decided before $\overline{P}$ is decided. In reality, applying the microwave once to the electrons means that the central position of the microwave spectrum is determined initially. It should be decided before $\overline{P}$ is decided, on the assumption that the frequency width of the microwave spectrum will be small. In this case, the dependency of the cooling rate on $\overline{P}$ is clearly different from the previous results. The cooling rate of Eq. (104) is expressed as

$$\nu = -2\log_{10}eS_0\frac{\hbar k_0}{p_T}\frac{\overline{P}^2}{1+\overline{P}^2}\frac{1}{\overline{P}/2\overline{Q}} \times\left[1-\frac{1-n_{01}}{1+n_{01}}\frac{1+\exp(-\beta\hbar\Omega)}{1-\exp(-\beta\hbar\Omega)}\right], \tag{105}$$

where $\overline{P}/2\overline{Q}$ is a variable instead of $\overline{Q}$. As seen in Eq. (105), the cooling rate for the initially chosen $\overline{P}/2\overline{Q}$ has no maximum. As $\overline{P}$ goes to infinity, the $\overline{P}$-dependent term of ν goes to 1. This implies that the cooling rate is almost constant for a large $\overline{P}$. In other words, we do not have to consider the $\overline{P}$-dependency of the cooling rate seriously. However, a larger $\overline{P}$ implies a larger microwave intensity, as shown in Eq. (95). The stationary temperature in Eq. (102) is

$$k_BT_{\parallel}^{\infty} = \frac{\left[\left(\dfrac{1+n_{01}}{1-n_{01}}\right)^2-1\right]}{\left[\dfrac{1+n_{01}}{1-n_{01}}-\dfrac{1+\exp(-\beta\hbar\Omega)}{1-\exp(-\beta\hbar\Omega)}\right]}\frac{1+\overline{P}^2}{4\overline{P}^2}\frac{\overline{P}}{2\overline{Q}}\frac{\hbar k_0}{m}p_T, \tag{106}$$

which shows that the $\overline{P}$-dependency and the $\overline{P}/2\overline{Q}$-dependency are complete inversions of those of ν. Therefore, the smaller $\overline{P}/2\overline{Q}$ gives the higher cooling rate and the lower stationary temperature.

IX. RESULTS OF MICROWAVE COOLING

Setting appropriate conditions for the initial state, the longitudinal temperature can be estimated after the state reaches an equilibrium. Our system is supposed to be immersed in a liquid helium heat bath, so that the initial temperatures (transverse and longitudinal) are 4.2 K, the temperature of liquid helium. We apply 10 T as the magnetic field, $0.7\times10^9/\mathrm{cm}^3$ as the number density of the plasma in the trap, and at least 10^4 as the Q factor of the microwave cavity. For the number density ($0.7\times10^9/\mathrm{cm}^3$), the critical temperature which gives $\Gamma_{\parallel}=170$ is approximately 14 mK.

The cooling rates of the plasma are calculated for various values of $\overline{Q}$, $\overline{P}$, and the normalized microwave intensity E_s should be determined by a constraint to keep the two level assumption. Given 10 T as the magnetic field strength and 4.2 K as the initial temperature, the transverse temperature should be lower than 12.0 K for over 90% of the particles to be in the ground and the first excited state. More generally, the temperature is determined from the rescaling condition mentioned previously. The temperature with the condition is

$$T_{\perp} \leqslant 12.0\times\frac{B}{10}, \tag{107}$$

for general B field in tesla unit.

Now the cooling rate ν, defined by Eq. (82), can be calculated as a function of $\overline{Q}$ and $\overline{P}/2\overline{Q}$, with a constant B-field and a function of E_s from the constraint. One of the results, based on the fact that the plasma profile is a Gaussian, is shown in Table I. As the table shows, the cooling rates have maximums at $\overline{P}/2\overline{Q}=2.0$ for all $\overline{Q}$s. In Table II, the microwave intensities which give the constraint that the transverse

TABLE I. The linear cooling rates $[\nu(\mathrm{h}^{-1})]$ for various values of $\overline{Q}$ and $\overline{P}$ are calculated from $\nu\approx-\log_{10}(T_{\parallel}/T_0)$.

	$\overline{P}/2\overline{Q}$					
$\overline{Q}$	0.5	1.0	1.5	2.0	2.5	3.0
1.0	0.24472	0.43393	0.56351	0.60289	0.55839	0.48058
2.0	0.28735	0.51529	0.66927	0.69143	0.60894	0.50475
3.0	0.30080	0.54520	0.70563	0.71653	0.62080	0.50980
4.0	0.30715	0.55991	0.72206	0.72659	0.62522	0.51161
5.0	0.31075	0.56825	0.73072	0.73152	0.62732	0.51246
6.0	0.31301	0.57340	0.73477	0.73529	0.62847	0.51293
7.0	0.31454	0.57679	0.73546	0.73598	0.62917	0.51321
8.0	0.31561	0.57914	0.73609	0.73710	0.62963	0.51339
9.0	0.31639	0.58082	0.73669	0.73787	0.62994	0.51352
10.0	0.31698	0.58206	0.73727	0.73842	0.63017	0.51361

TABLE II. The microwave intensities (E_s) for various values of $\overline{Q}$ and $\overline{P}$, are calculated from $E_s=(I\sigma_0/\hbar\Omega)/S_0$.

	$\overline{P}/2\overline{Q}$					
$\overline{Q}$	0.5	1.0	1.5	2.0	2.5	3.0
1.0	0.26535	0.33213	0.51747	0.90984	1.60143	2.60313
2.0	0.49788	0.69645	1.31880	2.85801	5.73339	9.87276
3.0	0.78576	1.18737	2.52114	6.01611	12.5850	21.9779
4.0	1.13730	1.81743	4.15398	10.4150	22.1709	38.9235
5.0	1.55790	2.59575	6.23124	16.0631	34.4937	60.7101
6.0	2.05140	3.52833	8.75931	22.9631	49.5540	87.3381
7.0	2.62074	4.61892	11.7413	31.1160	67.3521	118.808
8.0	3.26799	5.86992	15.1786	40.5225	87.8883	155.119
9.0	3.99462	7.28292	19.0722	51.1827	111.163	196.270
10.0	4.80183	8.85852	23.4227	63.0966	137.175	242.264

temperature is constant are shown for the sets of $(\overline{Q},\overline{P}/2\overline{Q})$. The smaller the number of electrons that interact with the microwave, the larger the microwave intensity that is required to keep the same transition rate. As Table II shows, as the central position of the microwave spectrum is farther away from the center of the profile, and as the frequency width of the microwave spectrum is smaller, the intensity in the constraint is larger. From the results shown in Table I and Table II, we can draw provisional conclusions about the Gaussian plasma profile transition. By applying a microwave of larger intensity and smaller frequency width to the Gaussian plasma profile at $p_z=\pm 2p_T$ of the profile, a larger transition rate can be obtained.

However, this conclusion does not apply to a long time scale. A microwave of large intensity and small frequency width initially creates a large transition between the two levels. At the same time, the electrons within the frequency width of the microwave spectrum lose their longitudinal momenta, so that they escape from the range of the microwave. After that, the number of electrons interacting with the microwave rapidly decreases, and there is almost no more microwave cooling, because of the large intensity and small width. Even worse, the small frequency width of the microwave spectrum causes the high speed electrons of the plasma to remain in their original positions over a long time scale, preventing the longitudinal temperature of the plasma from decreasing. In order to create microwave cooling over a long time scale, we have to apply a microwave of a large frequency width to the plasma. In this case, the intensity of the microwave should be small. Even though this does not create the fastest cooling initially, it may cause the fastest cooling over a long time scale. The other fact we have to consider during a simulation is that the cooling rate depends not only on $(\overline{Q},\overline{P}/2\overline{Q})$, but also on the shape of the profile. This means that the deformed profile as a sum of the ground and the first excited state, as functions of time, is generally far from a Gaussian and not predictable with only Eq. (60a) and Eq. (60b). The profiles as functions of time can be determined by the Fokker-Planck equation, Eq. (61), using the two equations Eq. (60a) and Eq. (60b). As seen in Eq. (82), one of the parameters of the cooling rate is the longitudinal temperature, which can generally be determined from the longitudinal kinetic energy. If the profile as a function of time had the same shape as the initial profile, we would not need to apply the constraint for E_s again after applying it initially, because the initial constraint with the dimensionless Fokker-Planck equation gives a general condition of E_s. Since the profile during a simulation is generally not a Gaussian, we should apply the constraint to the equation for each time step during the simulation. Changing the parameter causes different results for the simulation.

As a result of these considerations, we will now choose a set of parameters as an initial value of the microwave. As shown in Table I, the cooling rate is maximized at $\overline{P}/2\overline{Q}=2.0$ for all $\overline{Q}$s. From this result, we take (2.0,2.0) as the set of parameters $(\overline{Q},\overline{P}/2\overline{Q})$. As explained before, $\overline{P}/2\overline{Q}=2.0$, representing the central position of the microwave in velocity space, means that the center of the microwave is at $p=2p_T$ in velocity space. Also, $\overline{Q}=2.0$, representing the width of the microwave in velocity space, means that the width of the microwave is $\gamma/2p_T$ in velocity space. With this set of parameters, the microwave interacts initially with the electrons in the range from $p\approx 1.5p_T$ to $p\approx 2.5p_T$. Since the portion of the electrons outside the range, $p>2.5p_T$, is less than 1% of the electrons in velocity space, the microwave causes enough cooling to leave almost no high speed electrons, so cooling can be continued by changing the intensity of the microwave. If the portion outside the range is not that small, then high speed electrons will remain in the range, preventing the electrons of the entire profile from being cooled over a long time scale.

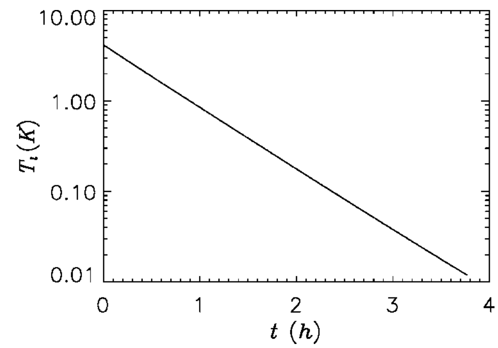

FIG. 1. Time evolution of the longitudinal temperature for $\overline{Q}=2.0$ and $\overline{P}/2\overline{Q}=2.0(N_1/N_0=0.2)$.

The microwave intensity has to be decided by the condition of the constant ratio between the ground and the first excited state. As the ratio is changed, the cooling rate should be changed. Let us consider two cases for the ratio. The first case is that the ratio, N_1/N_0, during a simulation, is 0.2, which means that only 4% of electrons are in the second or the higher excited states. The other case is that the ratio is 0.3, which means that 9% of electrons are in the second or the higher excited states. In order to investigate the basic behavior of the longitudinal temperature, let us apply $\overline{Q}=2.0$, $\overline{P}/2\overline{Q}=2.0$, and $N_1/N_0=0.2$ to our simulation. In Fig. 1 and Fig. 2, the longitudinal cooling and the microwave intensity required for $N_1/N_0=0.2$ are shown. As explained previously, the asymptotic behavior of the longitudinal temperature is clearly exponential. Actually, in Fig. 1, the entire

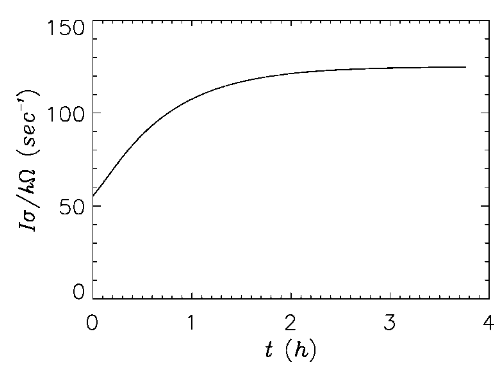

FIG. 2. The microwave intensity for $N_1/N_0=0.2$ when $\overline{Q}=2.0$ and $\overline{P}/2\overline{Q}=2.0$ is applied.

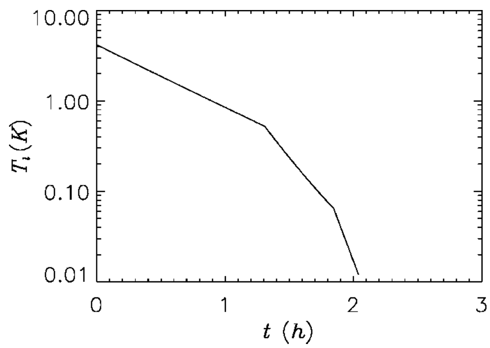

FIG. 3. Time evolution of the longitudinal temperature for $N_1/N_0=0.2$, applying the best cooling parameters whenever the profile starts asymptotic behavior.

behavior of the longitudinal temperature looks exponential. The exponent of our simulation is the same as the result of Eq. (105), where we apply $\overline{Q}=2.0$, $\overline{P}/2\overline{Q}=2.0$, and $n_{01}=0.2$. Figure 2 shows that the microwave intensity is saturated asymptotically. Even though the entire behavior of the longitudinal temperature in Fig. 1 is exponential, the microwave intensity is not the same as the saturated intensity initially. This means that the asymptotic assumption discussed previously cannot be applied to this system initially. In Fig. 2, we know that the asymptotic behavior starts approximately 1.5 h after the microwave is applied. The saturated intensity is also the same as the result of Eq. (95).

With the values of $\overline{Q}$, $\overline{P}/2\overline{Q}$, and $I\sigma_0$s determined from the condition of the constant ratio, it still takes too long to reach the critical temperature. From the result of our simulation, the cooling time is almost 4 h. This time is unrealistic in experiments, as a plasma profile in a Penning trap with a high magnetic field cannot stand for such a long time. The reason it takes too long is that the plasma profile does not interact effectively with the microwave after a certain time. The plasma is cooled so rapidly that the temperature reaches the value which breaks the condition in Eq. (87), and then the cooling becomes much slower, because the high peaked central frequency of the applied microwave is too far from the plasma profile. The real intensity of the microwave applied to the plasma is too weak to cool it as rapidly as it did before its temperature reached that value.

For this reason, we need to find another way to cool the plasma more rapidly. At moments when the plasma breaks

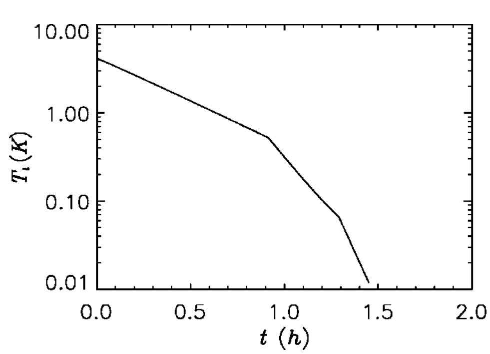

FIG. 4. Time evolution of the longitudinal temperature for $N_1/N_0=0.3$.

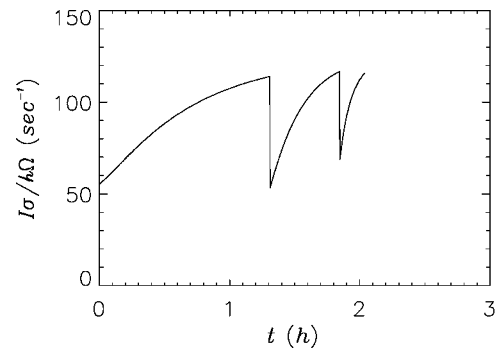

FIG. 5. Time evolution of the microwave intensity required for $N_1/N_0=0.2$ with the best cooling parameters.

the Eq. (87), we might change the central frequency and the frequency width of the microwave spectrum in momentum space. This could be achieved by changing $\overline{Q}$ and $\overline{P}/2\overline{Q}$. However, according to our results, the best way to get cooling is not to change $\overline{Q}$ and $\overline{P}/2\overline{Q}$ during the whole simulation, but to change $I\sigma_0$ at moments when the plasma breaks the condition.

Because the microwave pushes the electrons into the range of low longitudinal temperature, the initial Gaussian profile will have a distortion in the range where the electrons interact with the microwave. The cooling will be continued until there are no more electrons in that range, and most of electrons will move to the lower temperature range. In that range, the microwave intensity is so small that the microwave is almost linear to the velocity. The linear function causes a linear dissipative coefficient and a constant diffusion coefficient. With the aid of results from the classical Fokker-Planck equation, we know that the electron profile from the two coefficients will be Gaussian-like. This means that the profile will be almost a contraction of the initial profile. Therefore, even though γ and $\Delta\omega$ will be changed whenever the plasma breaks Eq. (87), $\overline{Q}$ and $\overline{P}/2\overline{Q}$ will not change from their initial values.

In Fig. 3 and Fig. 4, time evolutions are shown for the two sets of $(\overline{Q},\overline{P}/2\overline{Q})$ and the microwave intensities. Also, in Fig. 5 and Fig. 6, the intensities required for the condition of a constant ratio are shown. In Table III, the times when the parameters should be changed are determined. As we expected, the cooling times can be reduced as much as we need

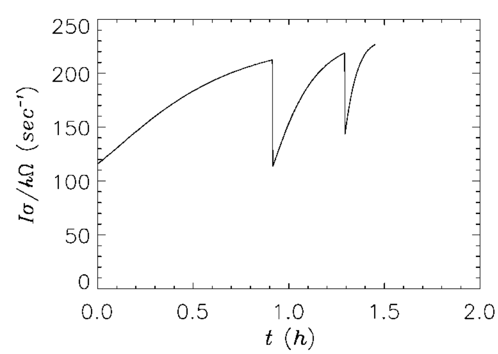

FIG. 6. Time evolution of the microwave intensity required for $N_1/N_0=0.3$.

TABLE III. The times when parameters should be changed.

t (sec)	$N_1/N_0=0.2$		
	$\alpha(\mathrm{h}^{-1})$	$T_\parallel$	$T_\perp$
0	0.69143	4.2	4.2
4720	1.68898	0.516901	8.3517
6650	4.12583	0.063347	8.3517
t (sec)	$N_1/N_0=0.3$		
	$\alpha(\mathrm{h}^{-1})$	$T_\parallel$	$T_\perp$
0	0.98948	4.2	4.2
3300	2.40939	0.515117	11.1644
4660	5.84879	0.063351	11.1644

them to be. The cooling time is more than 1 h and less than 2 h for $N_1/N_0=0.3$, and is slightly more than 2 h for $N_1/N_0=0.2$. Both times are very realistic for Penning trap experiments.

X. CONCLUSION

We have surveyed the molecular-dynamics simulation results of a strongly magnetized plasma, and concluded that crystallization can be achieved below a longitudinal critical temperature irrespective of transverse temperature. Applying a microwave in the same direction as the strong magnetic field line, we were finally able to create a cold plasma with a longitudinal coupling parameter over the critical value. On the basis of a small ratio between the ground and the first excited state, two level transition equations between the two levels were derived. A Fokker-Planck equation was derived from these two level transition equations. Also, we found a rescaling equation for the cooling rate, the results of which showed that the cooling time is proportional to the cubic of the rescaling parameter, as the magnetic field, the microwave intensity, and the transverse temperature are rescaled by the same rescaling parameter. When a microwave is applied in the longitudinal direction, the temperature can be decreased below the critical temperature by exchange of energy between the two degrees of freedom. The results of our simulations show that electron plasma crystallization can be achieved in a Penning trap in two hours.

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